ON ASYMPTOTIC ASSOUAD-NAGATA DIMENSION

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Abstract. For a large class of metric spaces X including discrete groups we prove that the asymptotic Assouad-Nagata dimension AN-asdim X of X coincides with the covering dimension $\dim(\nu_L X)$ of the Higson corona of X with respect to the sublinear coarse structure on X. Then we apply this fact to prove the equality AN-asdim$(X \times \mathbf{R}) =$ AN-asdim $X + 1$. We note that the similar equality for Gromov's asymptotic dimension asdim generally fails to hold [Dr3].

Additionally we construct an injective map $\xi : cone_\omega(X) \setminus [x_0] \to \nu_L X$ from the asymptotic cone without the base point to the sublinear Higson corona.

§1 Introduction

The Assouad-Nagata dimension was introduced in the 80s by Assouad [As1],[As2] under the name Nagata dimension. Recently this notion was revived in the asymptotic geometry due to works of Lang and Schlichenmaier [LSch], and Buyalo and Lebedeva [Bu], [BL]. The concept takes into account the dimension of a metric space on all scales. In this paper we consider only the large scale version of it. Note that the asymptotic version of the Assouad-Nagata dimension agrees with the original for our main source of examples of metric spaces - finitely generated discrete groups with the word metric. Like in the case of Gromov's asymptotic dimension, the Assouad-Nagata dimension is a group invariant.

A certain analogy between the asymptotic Assouad-Nagata dimension AN-asdim and the asymptotic dimension asdim invites one to transfer the asymptotic dimension theory [Gr],[Dr1],[Dr2],[Dr3],[DKU],[BD1],[BD2],[BD3], [DZ],[Ro2] to the asymptotic Assouad-Nagata dimension. It was partially done in [LSch], [BDHM], [BDLM]. Namely, the

1991 *Mathematics Subject Classification*. Primary 51F99, 54F45, Secondary 20H15.

Key words and phrases. dimension, asymptotic dimension, Assouad-Nagata dimension, Higson corona.

The first author was partially supported by an NSF grant. Also he would like to thank the Max-Planck Institute für Mathematik for hospitality.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

theorem on embedding into a product of trees, the characterization of asdim in terms of map extension, the union theorems, and the Hurewicz type theorem were successfully extended to the case of the Assouad-Nagata dimension. In this paper we extend to the Assouad-Nagata dimension the theorem that characterizes the asymptotic dimension of a metric space $\operatorname{asdim} X$ as the covering dimension of the Higson corona $\dim \nu X$ ([Dr1],[DKU]). For that we introduce a coarse structure $\mathcal{E}_L$ on a metric space X called the sublinear coarse structure and show that the covering dimension of the Higson corona $\nu_L X$ of this coarse structure is exactly the asymptotic Assouad-Nagata dimension of the space, provided the latter is finite. Contrary to the case of the classic Higson corona, the sublinear Higson corona ν_L behaves nicely under the product with reals. Namely, there is a decomposition: $\nu_L(X \times \mathbf{R}) = \nu_L X \times (-1,1) \cup \nu_L \mathbf{R}$. In particular, it implies $\dim \nu_L(X \times \mathbf{R}) = \dim \nu_L X + 1$. We prove that $\text{AN-asdim}\, X = \dim \nu_L X$ for sufficiently symmetric spaces X like discrete groups (it is not true in general). Then for such spaces $\text{AN-asdim}(X \times \mathbf{R}) = \text{AN-asdim}\, X + 1$. This is an analog of the classical Morita formula from the dimension theory: $\dim(X \times \mathbf{R}) = \dim X + 1$. We note that the Morita formula generally does not hold for asymptotic dimension [Dr3].

Coarse structures. A coarse structure $\mathcal{C}$ on a set X is a family of subsets $E \subset X \times X$ that contains the diagonal Δ_X and is closed taking finite unions, subsets, inverses, and compositions. The elements of $\mathcal{C}$ are called *controlled* sets (see [HR],[Ro2],[DH]).

Suppose that X is a topological space. Then a set $E \subset X \times X$ is called *proper* if both $E[K]$ and $E^{-1}[K]$ are relatively compact for a relatively compact set $K \subset X$, where $E[K]$ is the set of all x' such that there is $x \in K$ with $(x', x) \in E$. We use the notations $E_x = E[x]$ and $E^x = E^{-1}[x]$ for $x \in X$.

A subset $B \subset X$ of a coarse space is *bounded* if $B \times B$ is controlled. A map between coarse spaces $f : (X, \mathcal{C}) \to (x', \mathcal{C}')$ is called a *proper* if the preimage $f^{-1}(B)$ of every bounded set is bounded. A map between coarse spaces $f : (X, \mathcal{C}) \to (x', \mathcal{C}')$ is called a *coarse morphism* if it is coarsely proper and $(f \times f)$ takes controlled sets to controlled.

Suppose that X is a topological space. We say that a coarse space $(X, \mathcal{E})$ is *consistent* with the topology on X if $B \subset X$ is (coarsely) bounded if and only if B is relatively compact (i.e., bounded sets coincide with relatively compact sets). One can easily show a consistent coarse space $(X, \mathcal{E})$ is coarsely connected and each $E \in \mathcal{E}$ is proper. If X is a locally compact Hausdorff topological space, we say that $(X, \mathcal{E})$ is *proper* if $(X, \mathcal{E})$ is consistent with the topology and if $\mathcal{E}$ contains a neighborhood of the diagonal.

Compactifications. Let $\bar{X}$ be a compactification of a locally compact space X, and let V be an open subset of X. Then there is a unique maximal open set $\widetilde{V}$ in $\bar{X}$ such that $\widetilde{V} \cap X = V$. In fact, $\widetilde{V} = \bar{X} \setminus \overline{X \setminus V}$. One can show that $\widetilde{V} \subset \overline{V}$.

The following propositions are obvious.

Proposition 1.1. *Let $\bar{X}$ be a compactification of a locally compact space X, and let $\nu X = \bar{X} \setminus X$. Then $\{\widetilde{V} \cap \nu X : V$ is open in $X\}$ forms a basis for νX.*

Proposition 1.2. *Let $\bar{X}$ be a compactification of a locally compact space X, and let $\nu X = \bar{X} \setminus X$. Suppose $U \subset X$ is open and suppose $x \in \widetilde{U} \cap \nu X$. Then there is a set $V \subset U$ open in X such that $x \in \widetilde{V} \cap \nu X$ and $\bar{V} \subset \widetilde{U}$.*

Proof. Let W be an open subset of $\overline{X}$ such that $x \in W \subset \overline{W} \subset \widetilde{U}$, and set $V = W \cap X$. We have that V is open in X, $V \subset W$, and $W \subset \widetilde{V}$ by definition. Thus, $x \in \widetilde{V} \cap \nu X$ and $\overline{V} \subset \overline{W} \subset \widetilde{U}$. This completes the proof. □

Proposition 1.3. *Let $\bar{X}$ be a compactification of a locally compact space X, and let $\nu X = \bar{X} \setminus X$. Suppose U is an open subset of νX and $x \in U$. Then there is a set V which is open in X, $x \in \widetilde{V} \cap \nu X$, and $\bar{V} \cap \nu X \subset U$.*

Proof. Choose W_1 open in $\overline{X}$ such that $U = W_1 \cap \nu X$ and take W_2 open in $\overline{X}$ such that $x \in W_2 \subset \overline{W_2} \subset W_1$. Set $V = W_2 \cap X$. Thus, $W_2 \subset \widetilde{V}$, hence $x \in W_2 \cap \nu X \subset \widetilde{V} \cap \nu X$. Also, $\overline{V} \cap \nu X \subset \overline{W_2} \cap \nu X \subset W_1 \cap \nu X = U$ since $V \subset W_2$. □

Suppose $\overline{X}$ is a compactification of the locally compact Hausdorff space X. Then $(X, \overline{X})$ will be called a compactified pair. Now suppose,in addition, that $\mathcal{E}$ is a coarse structure which is consistent with the topology on X. We say that $f : X \to \mathbf{C}$ is a *Higson function*, denoted $f \in C_h(X, \mathcal{E})$, if for every $E \in \mathcal{E}$ and every $\epsilon > 0$, there is a compact set K such that $|f(x) - f(y)| < \epsilon$ whenever $(y, x) \in E \setminus K \times K$. Then by the GNS theorem there is a compactification $h_{\mathcal{E}}X$ of X called the *Higson compactification* such that the algebra of Higson functions $C_h(X, \mathcal{E})$ is isomorphic to $C(h_{\mathcal{E}}X)$. We define $h(X, \mathcal{E}) = (X, h_{\mathcal{E}}X)$. The *Higson corona* is defined by $\nu_{\mathcal{E}}X = h_{\mathcal{E}}X \setminus X$.

On the other hand, suppose we have a compactified pair $(X, \overline{X})$. Let $\mathcal{E}_{\overline{X}}$ be those $E \subset X \times X$ for which $\overline{E} \setminus X \times X \subset \Delta_{\partial X}$, where $\partial X = \overline{X} \setminus X$, $\overline{E}$ denotes the closure of E in $\overline{X} \times \overline{X}$, and Δ_A denotes the diagonal in $A \times A$. Then $\mathcal{E}_{\overline{X}}$ is a coarse structure on X which is consistent with the topology on X. We will sometimes use the notation of Roe [Ro2], $t\overline{X}$, instead of $(X, \mathcal{E}_{\overline{X}})$.

The following generalizes a definition from [DKU].

DEFINITION. For a general coarse space $(X, \mathcal{E})$, a finite system $E_1, \ldots, E_n$ of subsets of X *diverges* if

$$\bigcap_{i=1}^{n} F[E_i]$$

is bounded for each $F \in \mathcal{E}$.

Theorem 1.4. *Let X be a locally compact Hausdorff space equipped with a coarse structure $\mathcal{E}$ that is consistent with the topology. For a finite system $E_1, \dots, E_n$ of subsets of X, if $\nu X \cap [\cap_{i=1}^n \overline{E_i}] = \varnothing$, then the system $E_1, \dots, E_n$ diverges.*

Proof. We let $\overline{X}$ denote the Higson compactification with respect to this coarse structure. Suppose that the system $E_1, E_2, \dots, E_n$ does not diverge; so there is a controlled set F such that $\cap_{i=1}^n F[E_i]$ is not bounded. Thus, for each compact subset K of X, there is an $x_K \in (\cap_{i=1}^n F[E_i]) \setminus K$. The collection of compact subsets of X, ordered by inclusion, forms a directed set. We denote it by $\mathcal{K}$. In particular, $\{x_K\}_K$ is a net. Since $\overline{X}$ is compact, there is a convergent subnet $\{x_{g(\lambda)}\}_\lambda$ (here, $g : \Lambda \to \mathcal{K}$ is an order-preserving map between directed sets such that $g(\Lambda)$ is cofinal in $\mathcal{K}$). Thus, for K compact, we have by cofinality that there is a λ_0 such that $g(\lambda) \supset K$ whenever $\lambda \geq \lambda_0$. Hence, for $\lambda \geq \lambda_0$, we have $x_{g(\lambda)} \in X \setminus g(\lambda) \subset X \setminus K$. As X is locally compact, this means that $x := \lim_\lambda x_{g(\lambda)} \in \nu X$.

Now fix $1 \leq i \leq n$. Then $x_{g(\lambda)} \in F[E_i]$ for all λ; for each λ, choose $y_\lambda \in E_i$ such that $(x_{g(\lambda)}, y_\lambda) \in F$. Since $F \in \mathcal{E}$, by Proposition 2.45 (a) of [Ro2], we have $F \in t\overline{X}$. Thus, since $x_{g(\lambda)} \to x \in \nu X$, we must have $y_\lambda \to x$. Thus, $x \in \overline{E}_i$ for each i and so $\nu X \cap [\cap_{i=1}^n \overline{E_i}] \neq \varnothing$. $\square$

The following theorem can be found in [Ro2].

Theorem 1.5. *Let X and Y be locally compact, Hausdorff spaces equipped with coarse structures $\mathcal{E}$ and $\mathcal{F}$, respectively, which are consistent with the topologies. If $f : X \to Y$ is a coarse, continuous map, then f extends to a continuous map $\overline{f} : h_{\mathcal{E}}X \to h_{\mathcal{F}}Y$ such that $\overline{f}(\nu X) \subset \nu Y$.*

In fact, writing $h(f)$ rather than $\overline{f}$, we have that h is a functor from the category of consistent coarse structures (with the topological space being locally compact) with continuous coarse maps as morphisms, to the category of compactified pairs with continuous maps preserving boundary as morphisms. Also, there is a functor ν from the category of proper coarse structures with coarse maps as morphisms, to the category of compact spaces with continuous maps as morphisms; it sends $(X, \mathcal{E})$ to $\nu_{\mathcal{E}}X$. For the latter, see proposition 2.41 of [Ro2].

Asymptotic dimension. We recall Gromov's definition of *asymptotic dimension* of a metric space [Gr].

DEFINITION. The asymptotic dimension of a metric space X does not exceed n, $\operatorname{asdim} X \leq n$, if for every $r > 0$ there are r-disjoint uniformly bounded families $\mathcal{U}^0, \dots, \mathcal{U}^n$ of subsets of X such that the union $\cup \mathcal{U}^i$ is a cover of X.

There are equivalent reformulations [Gr],[BD2]:

Proposition 1.6. *Let (X, d) be a metric space. The following are equivalent:*

1. $\operatorname{asdim} X \le n$*;*
2. *for every $\epsilon > 0$, there is $b > 0$ and an $\epsilon-$Lipschitz, b-cobounded map $p : X \to P$ to an $n-$dimensional uniform simplicial complex P;*
3. *for all $r \ge 0$, there is a uniformly bounded cover $\mathcal{U}$ of X such that the Lebesgue number $L(\mathcal{U}) \ge r$ and $\mathcal{U}$ has multiplicity $\le n + 1$.*

Here a map of a metric space to a simplicial complex $f : X \to P$ is called b-cobounded if $\operatorname{diam}(f^{-1}(\sigma)) \le b$ for every simplex $\sigma \subset P$. A uniform metric on a simplicial complex P is the restriction of the Euclidean metric from $\ell_2(P^{(0)})$, the Hilbert space spanned by the vertices of P, to $P \subset \ell_2(P^{(0)})$. By the multiplicity of a cover $\mathcal{U}$ of X we mean the minimum number m such that every intersection of $m + 1$ distinct elements of $\mathcal{U}$ is empty. We will sometimes denote this number by $\operatorname{mult} \mathcal{U}$.

DEFINITION. The *asymptotic Assouad-Nagata dimension* of a metric space X does not exceed n, $\text{AN-asdim}\, X \le n$, if there is a $c > 0$ and an $r_0 > 0$ such that for every $r \ge r_0$, there is a cover $\mathcal{U}$ of X such that $\operatorname{mesh} \mathcal{U} \le cr$, $L(\mathcal{U}) > r$, and $\mathcal{U}$ has multiplicity $\le n + 1$.

This has many aliases, including *asymptotic dimension of linear type* and *asymptotic dimension with Higson property*. For discrete metric spaces, in particular for discrete finitely generated groups, this definition coincides with the Assouad-Nagata dimension.

There are analogous reformulations:

Proposition 1.7. *Let (X, d) be a metric space. The following are equivalent:*

1. $\text{AN-asdim}\, X \le n$*;*
2. *there is a $C > 0$ and an $\epsilon_0 > 0$ such that for all $\epsilon \le \epsilon_0$ ($\epsilon > 0$), there is an $\epsilon-$Lipschitz, $C/\epsilon-$cobounded map $p : X \to P$ to an $n-$dimensional simplicial complex P;*
3. *there is a $C > 0$ and an $r_0 > 0$ such that, for all $r \ge r_0$, there are $r-$disjoint families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ (of subsets of X) such that $\mathcal{U} = \cup_i \mathcal{U}_i$ is a cover of X and* $\operatorname{mesh} \mathcal{U} \le Cr$*;*
4. *there is a $C > 0$ and an $r_0 > 0$ such that for all $r \ge r_0$, there is a cover $\mathcal{U}$ of X such that* $\operatorname{mesh} \mathcal{U} \le Cr$ *and $B_r(x)$ meets at most $n + 1$ elements of $\mathcal{U}$ for each $x \in X$.*

In (1),(3), and (4), we can take the covers to be open.

§2 The sublinear coarse structure

We consider a proper metric space (X, d) with basepoint x_0 and define $\|x\| = d(x, x_0)$. We will sometimes write B_r to indicate $B_r(x_0)$, the open ball of radius r centered at x_0.

DEFINITION. We define the *sublinear coarse structure*, denoted $\mathcal{E}_L$, on X as follows:

$$\mathcal{E}_L = \{E \subset X \times X : E \text{ proper}, \lim_{x\to\infty} \frac{\sup_{y\in E_x} d(y,x)}{\|x\|} = 0 = \lim_{x\to\infty} \frac{\sup_{y\in E^x} d(x,y)}{\|x\|}\}.$$

By the statement $\lim_{x\to\infty} \frac{\sup_{y\in E_x} d(y,x)}{\|x\|} = 0$, we mean that for each $\epsilon > 0$, there is a compact subset K of X containing x_0 (equivalently, an $r \geq 0$) such that

$$\frac{\sup_{y\in E_x} d(y,x)}{\|x\|} \leq \epsilon$$

for all $x \notin K$ (respectively, for all x with $\|x\| > r$). It would perhaps be better to think of this as $\lim_{\|x\|\to\infty}$. In the event that $E_x = \varnothing$, we define $\sup_{y\in E_x} d(y,x) = 0$. We leave to the reader to check that $\mathcal{E}_L$ is indeed a coarse structure and that it does not depend on the choice of basepoint. The Higson corona for the sublinear coarse structure on X will be denoted by $\nu_L X$. We will sometimes call this corona the sublinear Higson corona, to eliminate possible confusion with the usual Higson corona.

We recall that a mapping $f : X \to Y$ between metric spaces is said to be a *quasi-isometry* if there are numbers $\lambda > 0$, $C \geq 0$, and $D \geq 0$ such that

$$\frac{1}{\lambda} d(x,y) - C \leq d(f(x), f(y)) \leq \lambda d(x,y) + C$$

and every point of Y is within distance D of $\phi(X)$.

Proposition 2.1. *Let X and Y be proper metric spaces. If $f : X \to Y$ is a quasi-isometry, then it is a coarse equivalence with respect to the sublinear coarse structures.*

Proof. Fix a basepoint $x_0 \in X$; set $y_0 = f(x_0) \in Y$. Choose $\lambda > 0$ and $C \geq 0$ such that $\frac{1}{\lambda} d(x,y) - C \leq d(f(x), f(y)) \leq \lambda d(x,y) + C$. Let g be a quasi-isometry such that $f \circ g$ and $g \circ f$ are close to the respective identity functions. It is clear that f is proper and the image under f of a bounded set is bounded. Let E be a controlled set in the sublinear coarse structure on X. It is not hard to show that $f \times f(E)$ is proper.

Let $\epsilon > 0$ be given. There is a bounded $K' \subset X$ such that $\frac{\sup_{y\in E_x} d(y,x)}{\|x\|} \leq \frac{\epsilon}{4\lambda^2}$ whenever $x \notin K'$. Set $K = B(x_0, 2\lambda C) \cup B(x_0, \frac{4\lambda C}{\epsilon}) \cup K'$. Note that $f(K)$ is bounded. Suppose that $z \notin f(K)$. If $(f \times f)(E)_z = \varnothing$, then $\dfrac{\sup\{d(z', z) : z' \in (f \times f)(E)_z\}}{\|z\|} = 0 < \epsilon$ and we are finished. Now assume that $z' \in (f \times f)(E)_z$; so $z' = f(x')$ and $z = f(x)$ for some $x, x' \in X$ with $(x', x) \in E$. We have that $x \notin K$.

Then

$$\|f(x)\| \geq \frac{1}{\lambda}\|x\| - C = \frac{2\|x\| - 2\lambda C}{2\lambda} = \frac{\|x\|}{2\lambda} + \frac{\|x\| - 2\lambda C}{2\lambda} \geq \frac{\|x\|}{2\lambda}$$

and so

$$\frac{d(z',z)}{\|z\|} = \frac{d(f(x'),f(x))}{\|f(x)\|} \leq 2\lambda\frac{\lambda d(x',x)+C}{\|x\|} = 2\lambda^2\frac{d(x',x)}{\|x\|} + \frac{2\lambda C}{\|x\|} \leq \epsilon.$$

Thus, we have $\dfrac{\sup\{d(z',z) : z' \in (f\times f)(E)_z\}}{\|z\|} \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, $(f \times f)(E)$ is controlled and so f is a coarse map.

Similarly, since g is a quasi-isometry, it is a coarse map as well. Finally, it is clear that $f \circ g$ and $g \circ f$ are close to the corresponding identities when X and Y are equipped with the sublinear coarse structures. Thus, f is a coarse equivalence. □

Corollary 2.2. *The sublinear coarse structure $\mathcal{E}_L$ is well-defined on finitely generated groups, i.e., the sublinear coarse structure for a given group Γ is independent of the choices of the finite generating set and the basepoint. In particular, the asymptotic dimension* $\mathrm{asdim}(\Gamma, \mathcal{E}_L)$ *associated with $\mathcal{E}_L$ is a group invariant for finitely generated groups.*

Next we give a characterization of divergent systems for the sublinear coarse structure.

Lemma 2.3. *Let (X,d) be a proper metric space with basepoint x_0, endowed with the sublinear coarse structure $\mathcal{E}_L$. For a finite system $E_1,\dots,E_n$ of subsets of X, the following are equivalent.*

(1) $\nu_L X \cap [\cap_{i=1}^n \overline{E_i}] = \varnothing$;
(2) *the system $E_1,\dots,E_n$ diverges;*
(3) *there exist $c, r_0 > 0$ such that* $\max_{1\leq i\leq n} d(x,E_i) \geq c\|x\|$ *whenever* $\|x\| \geq r_0$.

Proof. That (1) implies (2) was shown earlier. We now prove that (2) implies (3). Assuming that (3) does not hold, then if we let m be a positive integer, and if we set $c = \frac{1}{4m}$ and $r_0 = 2m$, then there is an x_m such that $\|x_m\| \geq 2m$ yet

$$\max_{1\leq i\leq n} d(x_m, E_i) < \frac{1}{4m}\|x_m\|.$$

Thus, for each i, we have that $d(x_m,E_i) < \frac{1}{4m}\|x_m\|$, and so there is an $a_m^i \in E_i$ such that $d(x_m,a_m^i) < \frac{1}{4m}\|x_m\|$. We have $d(a_m^i,a_m^j) < \frac{1}{2m}\|x_m\|$. Also,

$$\|x_m\| \leq \|a_m^i\| + d(a_m^i,x_m) < \|a_m^i\| + \frac{1}{4m}\|x_m\|,$$

and hence $(1-\frac{1}{4m})\|x_m\| < \|a^i_m\|$ (all i). Since $\frac{1}{4m} < 1/2$, we have $\|a^i_m\| > \frac{1}{2}\|x_m\|$. Thus,

$$d(a^i_m, a^j_m) < \frac{1}{2m}\|x_m\| < \frac{1}{m}\|a^j_m\| \quad \text{and} \quad \|a^i_m\| > \frac{1}{2}\|x_m\| \geq m$$

for all $1 \leq i, j \leq n$. Take $F_{i,j} = \{(a^i_m, a^j_m) : m = 1, 2, \ldots\}$ for each $1 \leq i, j \leq n$. Fixing i, j, we temporarily set $G = F_{i,j}$ for convenience, and show that G is controlled. Since $\|a^i_m\| \to \infty$ and $\|a^j_m\| \to \infty$ as $m \to \infty$, it follows that G is proper. Now let $\epsilon > 0$ be given, and take M to be a positive integer for which $1/M < \epsilon$; set $K = \{a^j_m : 1 \leq m < M\}$. Suppose that $x \notin K$. If $G_x = \varnothing$, then by our convention we have $\frac{\sup_{y \in G_x} d(y,x)}{\|x\|} = 0$. If $y \in G_x$, then there is a positive integer m such that $(y, x) = (a^i_m, a^j_m)$, and since $a^j_m = x \notin K$, we must have $m \geq M$; it follows that $\frac{d(y,x)}{\|x\|} = \frac{d(a^i_m, a^j_m)}{\|a^j_m\|} < \frac{1}{m} < \epsilon$ and so $\frac{\sup_{y \in G_x} d(y,x)}{\|x\|} \leq \epsilon$. Thus, $\lim_{x\to\infty} \frac{\sup_{y \in G_x} d(y,x)}{\|x\|} = 0$. Similarly, $\lim_{x\to\infty} \frac{\sup_{y \in G^x} d(x,y)}{\|x\|} = 0$, and hence $G = F_{i,j}$ is controlled.

Define $F = \cup_{1 \leq j \leq n} F_{1,j}$ and $A = \{a^1_m : m = 1, 2, \ldots\}$. Note that A is not bounded and F is controlled. Also, $F[E_j] \supset F_{1,j}[E_j] \supset A$ for all $1 \leq j \leq n$ and hence

$$\cap_{j=1}^n F[E_j] \supset A,$$

which means that $\cap_{j=1}^n F[E_j]$ is not bounded. So (2) does not hold.

It remains to show that (3) implies (1). Define $F_i = E_i \setminus B_{r_0 + cr_0}$ for $1 \leq i \leq n$. Let $f : X \to \mathbf{R}$ be defined by $f(x) = \sum_{i=1}^n d(x, F_i)$. Note that $f(x) \geq c\|x\|$ when $\|x\| \geq r_0$ since $d(x, F_i) \geq d(x, E_i)$. Also, $f(x) \geq cr_0$ when $\|x\| \leq r_0$; in particular, $f(x) \geq c\|x\|$ for all x and $f(x) > 0$ for all x. Define $g_i : X \to \mathbf{R}$ by $g_i(x) = d(x, F_i)/f(x)$.

Let E be a controlled set. Since

$$|g_i(y) - g_i(x)| \leq d(y, F_i)|\frac{1}{f(y)} - \frac{1}{f(x)}| + |\frac{d(y, F_i) - d(x, F_i)}{f(x)}| \leq \frac{d(y, F_i)}{f(x)f(y)}|f(x) - f(y)|$$
$$+ \frac{d(y,x)}{f(x)} \leq \frac{nd(y,x)}{f(x)} + \frac{d(x,y)}{f(x)} \leq (n+1)\frac{d(x,y)}{c\|x\|},$$

we have $\sup_{y \in E_x} |g_i(x) - g_i(y)| \to 0$ as $x \to \infty$. Since E was an arbitrary controlled set, g_i (viewed as a map to $\mathbf{C}$) is a Higson function for each i. Let $G_i : \overline{X} \to \mathbf{C}$ be the extension of g_i to the Higson compactification. Since $\sum_i g_i = 1$, it is immediate that $\sum_i G_i = 1$ throughout $\overline{X}$. Also, $\overline{F_i} \subset G_i^{-1}(0)$ and it is not hard to see that $\nu X \cap \overline{F_i} = \nu X \cap \overline{E_i}$. Thus,

$$\nu X \cap (\cap_{i=1}^n \overline{E_i}) = \nu X \cap (\cap_{i=1}^n \overline{F_i}) \subset \nu X \cap (\cap_{i=1}^n G_i^{-1}(0)) = \varnothing$$

since $\sum_i G_i = 1$ on νX. $\square$

In the case that $n = 2$, we can add another condition.

Lemma 2.4. *Let A and B be subsets of a metric space X. Let $x_0 \in X$, and define $\|\cdot\|$ as usual. Also, take $B_r = B_r(x_0)$. Then the following are equivalent.*

1. *There exist $C, r_0 > 0$ such that* $\max\{d(x,A), d(x,B)\} \geq C\|x\|$ *whenever $\|x\| \geq r_0$;*
2. *there exist $D, r_1 > 0$ such that $d(A \setminus B_r, B \setminus B_r) \geq Dr$ whenever $r \geq r_1$.*

Proof. We show (1) implies (2). Given C and r_0, take $D = C$ and $r_1 = r_0$. Let $r \geq r_1$, $a \in A \setminus B_r$, and $b \in B \setminus B_r$. So $\|a\| \geq r \geq r_0$. Thus,

$$Cr \leq C\|a\| \leq \max\{d(a,A), d(a,B)\} = \max\{0, d(a,B)\} = d(a,B) \leq d(a,b)$$

by (1). So $d(A \setminus B_r, B \setminus B_r) \geq Dr$.

We show (2) implies (1). Let D, r_1 be positive numbers satisfying (2). Set $r_0 = 2r_1$ and take C to be a positive number satisfying $C < \min\{1/2, D/4\}$. Now let $x \in X$ be such that $\|x\| \geq r_0 = 2r_1$. To get a contradiction, suppose that $\max\{d(x,A), d(x,B)\} < C\|x\|$. Then $d(x,A) < C\|x\|$ and $d(x,B) < C\|x\|$. Thus, there exist $a \in A$ and $b \in B$ such that $d(x,a) < C\|x\|$ and $d(x,b) < C\|x\|$. So $d(a,b) < 2C\|x\|$. We then have

$$\|a\| \geq \|x\| - d(x,a) > \|x\| - C\|x\| = (1-C)\|x\| \geq \|x\|/2.$$

Similarly, $\|b\| \geq \|x\|/2$. Since $\|x\|/2 \geq r_1$, we have by (2) that

$$\frac{D\|x\|}{2} \leq d(A \setminus B_{\|x\|/2}, B \setminus B_{\|x\|/2}) \leq d(a,b) < 2C\|x\| \leq \frac{D\|x\|}{2},$$

a contradiction. Therefore, $\max\{d(x,A), d(x,B)\} \geq C\|x\|$ when $\|x\| \geq r_0$. □

DEFINITION. Let (X,d) be a metric space, and let $\mathcal{V}$ be a family of open subsets of X. We define the *Lebesgue function* associated with the cover $\mathcal{V}$, denoted $L^{\mathcal{V}}$, by

$$L^{\mathcal{V}}(x) = \sup_{V \in \mathcal{V}} d(x, X \setminus V).$$

DEFINITION. For a proper metric space (X,d) with basepoint x_0, we say a function $f : X \to [0,\infty)$ is (eventually) *at least linear* if there exist $c, r_0 > 0$ such that $f(x) \geq c\|x\|$ whenever $\|x\| \geq r_0$.

Corollary 2.5. *Let (X,d) be a proper metric space endowed with the coarse structure $\mathcal{E}_L$. Let $\alpha = \{O_1, \ldots, O_n\}$ be a finite family of open subsets of X. Then $\widetilde{\alpha} = \{\widetilde{O}_1, \ldots, \widetilde{O}_n\}$ covers the corona $\nu_L X$ if and only if the Lebesgue function L^α is at least linear.*

Proof. $\widetilde{\alpha} = \{\widetilde{O}_1, \ldots, \widetilde{O}_n\}$ covers the corona νX iff $\nu X \setminus (\cup_i \widetilde{O}_i) = \varnothing$, iff $\nu X \cap (\overline{X} \setminus \cup_i \widetilde{O}_i) = \varnothing$, iff $\nu X \cap (\cap_i (\overline{X} \setminus \widetilde{O}_i)) = \varnothing$, iff $\nu X \cap (\cap_i \overline{X \setminus O_i}) = \varnothing$ by the comments preceding Proposition 1.1, iff the system $X \setminus O_1, \ldots, X \setminus O_n$ diverges, which, by lemma 2.3 above, is true if and only if L^α is at least linear. □

Corollary 2.6. *Let (X, d) be a proper metric space, and let A be a closed subspace of X equipped with the restricted metric. Then the embedding $A \to X$ extends to an embedding $h_L A \to h_L X$ on the compactifications and induces an embedding $\nu_L A \to \nu_L X$ on the coronas.*

Proof. Let $x_0 \in A$ be the basepoint for both A and X, and write $hX = h_L X$ and $\nu X = \nu_L X$. Since the inclusion map $i : A \to X$ is continuous and coarse, i extends to a continuous map from hA to hX such that $i(\nu A) \subset \nu X$. To prove the result, it suffices to show that i is injective on νA. Let $x_1, x_2 \in \nu A$ with $x_1 \neq x_2$. First, we can find disjoint open subsets U_1, U_2 of νA such that $x_j \in U_j$ for $j = 1, 2$. Applying Proposition 1.3, there are open subsets V_1 and V_2 of A such that $x_j \in \widetilde{V}_j \cap \nu A$ and $\operatorname{cl}_{hA} V_j \cap \nu X \subset U_j$ for $j = 1, 2$. So $\nu A \cap \operatorname{cl}_{hA} V_1 \cap \operatorname{cl}_{hA} V_2 \subset U_1 \cap U_2 = \varnothing$, and by Lemmas 2.3 and 2.4, we have that there exist $c, r_0 > 0$ such that

$$d(V_1 \setminus B_r, V_2 \setminus B_r) = d|_A(V_1 \setminus B_r, V_2 \setminus B_r) \geq cr$$

whenever $r \geq r_0$. Thus, $\nu X \cap \operatorname{cl}_{hX} V_1 \cap \operatorname{cl}_{hX} V_2 = \varnothing$. But $i(x_j) \in i(\operatorname{cl}_{hA} V_j) \cap \nu X = \operatorname{cl}_{hX} V_j \cap \nu X$ for $j = 1, 2$, which means that $i(x_1) \neq i(x_2)$. $\square$

Algebra of functions. We define a subalgebra $U(X) = U(X, x_0)$ of $C(X)$ as follows: $f : X \to \mathbf{C}$ is in $U(X)$ if and only if f is bounded, continuous, and there exists a $c = c_f$ such that

$$|f(x) - f(y)|\|x\| \leq cd(x, y).$$

It is not difficult to check that $U(X)$ is closed under addition, multiplication, and complex conjugation.

REMARK. In the definition, the continuity condition is almost unnecessary. It is not hard to see that the property $|f(x) - f(y)|\|x\| \leq cd(x, y)$ implies that f is continuous for $x \neq x_0$.

It is easy to show that $U(X)$ separates points and closed sets. It is also clear that $U(X)$ is, in general, not complete. We set $C'(X) = \overline{U(X)}$, where the bar represents closure in $C(X)$ with the uniform metric. So $C'(X)$ is a C^*-algebra which separates points and closed sets. Thus, by the GNS Theorem we can extract a compactification of X which will be called the *sublinear compactification*:

Proposition 2.7. *With the notation above, there is a compactification $\overline{X}$ of X such that $C'(X) = C(\overline{X})$.*

Let $h_L X$ be the Higson compactification for $\mathcal{E}_L$, the sublinear coarse structure on X. We have the following.

Proposition 2.8. *We have $C'(X) \subset C_h(X, \mathcal{E}_L)$, and hence there is a surjective, continuous map $h_L X \to \overline{X}$ which extends the identity.*

Proof. Let $f \in U(X, x_0)$, and let c be a constant such that $|f(x) - f(y)|\|x\| \leq cd(x, y)$. Let $E \in \mathcal{E}_L$. So

$$\lim_{x\to\infty} \sup_{y\in E_x} |f(x) - f(y)| \leq c \lim_{x\to\infty} \frac{\sup_{y\in E_x} d(y, x)}{\|x\|} = 0.$$

But $E \in \mathcal{E}_L$ was arbitrary, so $f \in C_h(X, \mathcal{E}_L)$. $\square$

We prove that the map $h_L X \to \overline{X}$ is a homeomorphism.

Proposition 2.9. *Let A and B be subsets of a proper metric space (X, d), and suppose that there is a constant $c > 0$ such that $d(A \setminus B_r, B \setminus B_r) \geq cr$ for all $r \geq 0$. Then the function $\phi : X \to [0, 1]$, defined by*

$$\phi(x) = \frac{d(x, A)}{d(x, A) + d(x, B)},$$

is an element of $U(X, x_0)$, i.e. there is a $c_\phi > 0$ such that $|\phi(x) - \phi(y)|\|x\| \leq c_\phi d(x, y)$ for all $x, y \in X$.

Proof. By the proof (not just the statement) for the characterization of divergent systems with two members, there is a number $C > 0$ such that

$$d(x, A) + d(x, B) \geq \max\{d(x, A), d(x, B)\} \geq C\|x\|$$

for all x with $\|x\| \geq 0$, that is for all $x \in X$. So

$$|\phi(x) - \phi(y)| \leq \left|\frac{d(x, A)}{d(x, A) + d(x, B)} - \frac{d(y, A)}{d(x, A) + d(x, B)}\right| + \left|\frac{d(y, A)}{d(x, A) + d(x, B)} - \frac{d(y, A)}{d(y, A) + d(y, B)}\right|$$

$$\leq \frac{d(x, y)}{d(x, A) + d(x, B)} + \frac{d(y, A)}{d(y, A) + d(y, B)} \frac{|d(y, A) - d(x, A)| + |d(y, B) - d(x, B)|}{d(x, A) + d(x, B)}$$

$$\leq 3\frac{d(x, y)}{d(x, A) + d(x, B)} \leq \frac{3d(x, y)}{C\|x\|},$$

and the Proposition follows. $\square$

Proposition 2.10. *Let (X,d) be a proper metric space, and suppose that A and B are subsets of X. Define $\nu X = \overline{X} \setminus X$, and set $A' = \overline{A} \cap \nu X$ and $B' = \overline{B} \cap \nu X$. If there exist $c, r_0 > 0$ such that $d(A \setminus B_r, B \setminus B_r) \geq cr$ whenever $r \geq r_0$, then $A' \cap B' = \varnothing$.*

Proof. Set $E = A \setminus B_{r_0}$ and $F = B \setminus B_{r_0}$. So $d(E \setminus B_r, F \setminus B_r) = d((A \setminus B_{r_0}) \setminus B_r, (B \setminus B_{r_0}) \setminus B_r)$. Then for $r \geq r_0$, we have $d(E \setminus B_r, F \setminus B_r) = d(A \setminus B_r, B \setminus B_r) \geq cr$, while if $r \leq r_0$, then $d(E \setminus B_r, F \setminus B_r) = d(A \setminus B_{r_0}, B \setminus B_{r_0}) \geq cr_0 \geq cr$. In any case, we have

$$d(E \setminus B_r, F \setminus B_r) \geq cr \quad \text{for all} \quad r \geq 0.$$

By the previous Proposition, the map $\phi(x) = \frac{d(x,E)}{d(x,E)+d(x,F)}$ lies in $U(X, x_0)$; so ϕ extends to $\overline{X}$, and we label this extension ϕ as well. Let $a \in A'$ and $b \in B'$. So $(a,b) \in \overline{A} \times \overline{B}$, and hence there is a net $\{(a_\alpha, b_\alpha)\}_\alpha$ of points of $A \times B$ such that $a_\alpha \to a$ and $b_\alpha \to b$. Since $a_\alpha \notin B_{r_0}$ and $b_\alpha \notin B_{r_0}$ eventually, and hence $a_\alpha \in E$ and $b_\alpha \in F$ eventually, we have that $\phi(a) - \phi(b) = \lim_\alpha(\phi(a_\alpha) - \phi(b_\alpha)) = -1$. That is, $\phi(a) \neq \phi(b)$, so $a \neq b$. Since $a \in A'$ and $b \in B'$ were arbitrary, have $A' \cap B' = \varnothing$. $\square$

Theorem 2.11. *Let (X,d) be a proper metric space. Then the sublinear compactification $\overline{X}$ is homeomorphic to the Higson compactification $h_L X$ for the sublinear coarse structure $\mathcal{E}_L$ via a homeomorphism extending the identity on X.*

Proof. We use $\nu_L X$ to denote the Higson corona associated with the sublinear coarse structure; $\nu' X$ will denote the boundary of $\overline{X}$, that is, $\nu' X = \overline{X} \setminus X$.

Let $\theta : h_L X \to \overline{X}$ be this extension of id_X. One can show that $\theta(\nu_L X) \subset \nu' X$. Thus, is suffices to show that the map is one-to-one on the corona. So let x and y be distinct points of $\nu_L X$. So there are subsets A and B of X such that $x \in A'$, $y \in B'$, and $\overline{A} \cap \overline{B} \cap \nu_L X = \varnothing$ (here the closure is taken in $h_L X$). This means that $d(A \setminus B_r, B \setminus B_r) \geq cr$ eventually (for some c). Thus, by Proposition 2.10, $\overline{A} \cap \overline{B} \cap \nu' X = \varnothing$ (closures take place in $\overline{X}$). But $\theta(x) \in \overline{A} \cap \nu' X$ and $\theta(y) \in \overline{B} \cap \nu' X$, so $\theta(x) \neq \theta(y)$. $\square$

REMARK. We have $C_h(X, \mathcal{E}_L) = \overline{U(X, x_0)}$. We will sometimes refer to a function $f \in C_h(X, \mathcal{E}_L)$ as a linear Higson function on X.

§3 The equality AN-asdim X = $\dim \nu_L X$

Asymptotic Assouad-Nagata dimension. We characterize the asymptotic Assouad-Nagata dimension using a sequential formulation.

For a map $f : X \to Y$ between metric spaces, define

$$\mathrm{Lip}\,(f) = \sup\{\frac{d_Y(f(x), f(y))}{d_X(x,y)} : x, y \in X, x \neq y\}.$$

Note that this could be ∞.

For a map $p : X \to P$ to a uniform simplicial complex P and a simplex $\Delta \subset P$, define

$$D(p, \Delta) = \inf \{\mathrm{Lip}\,(g) \mid g : p^{-1}(\Delta) \to \partial\Delta \ \text{ continuous}, g|_{p^{-1}\partial\Delta} = p|_{p^{-1}\partial\Delta}\}.$$

It might be possible that there are no such g with this property, or that $\mathrm{Lip}(g) = \infty$ for all such g, in which case $D(p, \Delta) = \infty$. Finally, we define

$$D_n(p) = \sup \{D(p, \Delta) : \Delta \subset P, \dim \Delta = n\}.$$

Based on the previous observations, this could be infinite as well. In the event that P is $n-$dimensional (the very case where we intend to use these constructions), we will write $D(p)$ rather than $D_n(p)$.

The following lemma will be crucial for this section.

Lemma 3.1. *For a metric space (X, d), suppose that* AN-asdim $X = n \geq 1$. *Then there is a sequence $\{\lambda_m\}_{m=1}^{\infty}$ of positive numbers, a sequence $\{p_m : X \to P_m\}_m$ of maps to $n-$dimensional simlicial complexes P_m, and a number $C > 0$ such that*

1. $\lambda_m \to 0$ *as* $m \to \infty$;
2. p_m *is* λ_m-*Lipschitz and* C/λ_m-*cobounded;*
3. $\displaystyle\lim_{m\to\infty} \frac{D(p_m)}{\lambda_m} = \infty.$

Proof. Set $a = (2n + 3)^2$ for simplicity. There is a $C > 0$, a positive integer m_0, and (for the sequence $\{am\}_{m\geq m_0}$) a sequence of covers $\mathcal{U}_m$ such that $L(\mathcal{U}_m) > am$, $\mathrm{mesh}\,\mathcal{U}_m \leq Cam$, and each ball $B_{am}(x)$ meets at most $n + 1$ elements of $\mathcal{U}_m$ for each $m \geq m_0$.

Let $p_m : X \to \mathrm{Nerve}\,\mathcal{U}_m$ be the projection to the nerve. It follows that p_m is $\frac{1}{m}-$Lipschitz and $Cam-$cobounded [BD2]. Set $\lambda_m = \frac{1}{m}$ and $P_m = \mathrm{Nerve}\,\mathcal{U}_m$.

We now show that $\limsup_{m\geq m_0} \frac{D(p_m)}{\lambda_m} = \infty$. To get a contradiction, suppose that there is a positive integer $m_1 \geq m_0$ and a $b > 1$ such that $\frac{D(p_m)}{\lambda_m} < b$ for all $m \geq m_1$. Fix $m \geq m_1$, and let Δ be an $n-$simplex in P_m. Thus, $D(p_m, \Delta) < b\lambda_m$, and so there is a $g_\Delta : p_m^{-1}(\Delta) \to \partial\Delta$ (depending on m) such that $\mathrm{Lip}(g_\Delta) < b\lambda_m$ and $g_\Delta = p_m$ on $p_m^{-1}(\partial\Delta)$. For $m \geq m_1$, we define a map $q_m : X \to (P_m)^{n-1}$ by

$$q_m(x) = \begin{cases} g_\Delta(x) & \text{if } \ x \in p_m^{-1}\Delta \text{ for some } n-\text{dimensional simplex } \ \Delta \subset P_m \\ p_m(x) & \text{otherwise.} \end{cases}$$

Here $(P_m)^{n-1}$ denotes the $n - 1$ skeleton of P_m. It is easy to see that q_m is well-defined and $q_m = p_m$ on $p_m^{-1}((P_m)^{n-1})$.

For $m \geq m_1$, define $\mathcal{V}_m = \{q_m^{-1}(\operatorname{st} v) : v$ is a vertex of $P_m\}$. We show $\operatorname{mesh} \mathcal{V}_m \leq Cam$. It suffices to show that $q_m^{-1}(\operatorname{st} v) \subset p_m^{-1}(\operatorname{st} v)$, where v is a vertex of $(P_m)^{n-1}$ (the vertices of $(P_m)^{n-1}$ are the same of those of P_m, and can be identified with the elements of the cover $\mathcal{U}_m$). Let $x \in q_m^{-1}(\operatorname{st} v)$, and we consider two cases. First, suppose that $x \in p_m^{-1}((P_m)^{n-1})$; then by definition, we have $p_m(x) = q_m(x) \in \operatorname{st} v$, and so $x \in p_m^{-1}(\operatorname{st} v)$. Second, if $x \in p_m^{-1}(\operatorname{int}\Delta)$ for some $n-$dimensional Δ, then $q_m(x) \in \partial\Delta$ and since $q_m(x) \in \operatorname{st} v$, we have v is a vertex of Δ; as $p_m(x) \in \operatorname{int}\Delta$, we have $p_m(x) \in \operatorname{st} v$, and so $x \in p_m^{-1} \operatorname{st} v$. This proves the inclusion.

We now show that $\mathcal{V}_m$ is $\frac{m}{b(n+1)}-$Lipschitz. Let $x \in X$. Let $U_0, U_1, \dots, U_j$ be the distinct members of $\mathcal{U}_m$ meeting $B(x, \frac{m}{b(n+1)})$. By the construction of $\mathcal{U}_m$, we must have $j \leq n$. Let $y \in B(x, \frac{m}{b(n+1)})$. Note that $p_m(x)$ (and $p_m(y)$ as well) lies in a simplex whose vertices form a subset of $\{U_i\}$. There is an i, $0 \leq i \leq j$, such that the U_i-th coordinate of $q_m(x)$ is at least $\frac{1}{n+1}$, i.e. $[q_m(x)]_{U_i} \geq \frac{1}{n+1}$. We consider two cases. First, suppose that $p_m(x)$ and $p_m(y)$ lie in the $(n-1)-$skeleton of P_m; that is, $x, y \in p_m^{-1}((P_m)^{n-1})$. Second, if $p_m(x)$ or $p_m(y)$ lies in the interior of an $n-$simplex, then we must have $j \geq n$, and so $j = n$. This means that $\{U_i : i = 0, 1, \dots, n\}$ corresponds to an $n-$simplex Δ of P_m, and so $x, y \in p_m^{-1}(\Delta)$. In either case,

$$|q_m(x) - q_m(y)| < b\lambda_m d(x,y) \leq \frac{1}{n+1},$$

and so $|[q_m(x)]_{U_i} - [q_m(y)]_{U_i}| < \frac{1}{n+1}$. Thus, $[q_m(y)]_{U_i} > 0$, or $y \in q_m^{-1}(\operatorname{st} U_i)$. This proves that $B(x, \frac{m}{b(n+1)}) \subset q_m^{-1}(\operatorname{st} U_i)$. As $x \in X$ was arbitrary, we have $L(\mathcal{V}_m) \geq \frac{m}{b(n+1)}$.

So for $m \geq m_1$, $\mathcal{V}_m$ is a cover of X, $\operatorname{mult} \mathcal{V}_m \leq n$, $L(\mathcal{V}_m) \geq \frac{1}{b(n+1)} m$, and $\operatorname{mesh} \mathcal{V}_m \leq Cam$. Now let r be a real number with $r \geq \frac{m_1}{b(n+1)}$, and choose an integer $m \geq m_1$ such that $m - 1 \leq b(n+1)r \leq m$. Then $L(\mathcal{V}_m) \geq \frac{m}{b(n+1)} \geq r$ and $\operatorname{mesh} \mathcal{V}_m \leq Cam \leq Ca(b(n+1)r + 1) \leq Ca(b(n+1)r + m_1) \leq 2Cab(n+1)r$. Setting $r_0 = \frac{m_1}{b(n+1)}$ and $C_0 = 2Cab(n+1)$ gives AN-asdim $X \leq n-1$, a contradiction.

Thus, $\limsup \frac{D(p_m)}{\lambda_m} = \infty$. Passing to a subsequence if necessary and relabeling, we have the desired result. $\square$

Extensions of functions. Let $A \subset X$ be a closed subset. We call a neighborhood $W \supset A$ *linear* if there is a constant $c > 0$ such that $d(A \setminus B_r, B \setminus B_r) \geq cr$ for all $r > 0$ where $B = X \setminus W$.

Suppose that $A_1 \subset A_2 \subset W$, where A_1 and A_2 are closed subsets of X and W is a linear neighborhood of A_2. Then W is a linear neighborhood of A_1 as well. Let

$A \subset W \subset X$, where W is a linear neighborhood in X of the closed set A. If $Y \subset X$ is a closed subset, then $Y \cap W$ is a linear neighborhood in Y of the closed set $Y \cap A$.

We shall extend the notation $U(X, x_0)$ as follows. For metric spaces X and Y (not necessarily proper), with $x_0 \in X$ and $\| \cdot \| = d_X(\cdot, x_0)$, we say $f \in U(X, x_0, Y)$ if and only if f is bounded, continuous, and there is a $c_f \geq 0$ such that $d_Y(f(x), f(y))\|x\| \leq c_f d_X(x, y)$ for all $x, y \in X$. In the event that $Y = \mathbf{C}$, we will omit $\mathbf{C}$ from the notation.

REMARKS. Suppose that $f \in U(X, x_0, Y)$ and that $g : Y \to Z$ is a $\lambda-$Lipschitz map between metric spaces. Then $d_Z((g \circ f)(x), (g \circ f)(y))\|x\| \leq \lambda c_f d_X(x, y)$, and so $g \circ f \in U(X, x_0, Z)$. It is also clear that $f \in U(X, x_0, \mathbf{R}^{n+1})$ if and only if $f_i \in U(X, x_0, \mathbf{R})$ for $1 \leq i \leq n+1$.

Proposition 3.2. *Let $g : X \to Y$ be an element of $U(X, x_0, Y)$ for proper metric spaces X and Y. Then $W = g^{-1}(N_r(F))$ is a linear neighborhood of $A = g^{-1}(F)$ for any closed subset $F \subset Y$ and $r > 0$.*

Proof. Let $c = c_g$. Assume that W is not linear. Then for each positive integer n, there is an $r_n > 0$ such that $d((X \setminus W) \setminus B_{r_n}, A \setminus B_{r_n}) < \frac{1}{n} r_n$, and so there are $x_n, y_n \in X$ such that $g(x_n) \in F$, $d(g(y_n), F) \geq r$, $\|x_n\|, \|y_n\| \geq r_n$, and $d(y_n, x_n) < \frac{1}{n} r_n$. Then we obtain a contradiction:

$$0 < r \leq |g(x_n) - g(y_n)| \leq \frac{c d(x_n, y_n)}{\|x_n\|} \leq c \frac{d(x_n, y_n)}{r_n} \leq \frac{c}{n} \to 0.$$

□

Proposition 3.3. *Let $q : X \to \mathbf{R}$ be a Higson function for the sublinear coarse structure for a proper metric space X. Then for every $\epsilon > 0$ there is a linear neighborhood $W \supset q^{-1}(0)$ such that $W \subset q^{-1}(-\epsilon, \epsilon)$.*

Proof. Since $q \in C(\overline{X})$, there is a function $g \in U(X, x_0)$ with $|g - q| < \epsilon/4$. Take $W = g^{-1}(-\epsilon/2, \epsilon/2) = g^{-1}(N_{\epsilon/4}([-\epsilon/4, \epsilon/4]))$. By Proposition 3.2 it is a linear neighborhood of $g^{-1}([-\epsilon/4, \epsilon/4] \supset q^{-1}(0)$. □

Proposition 3.4. *Let (X, d) be a proper metric space with basepoint x_0. Let A be a closed subset of X containing the basepoint, and let W be an open linear neighborhood of A. Suppose that $f \in U(W, x_0)$ and let $\bar{f}$ be an extension of $f|_A$ which is a linear Higson function on X. Then for every $\epsilon > 0$ there is a $g \in U(X, x_0)$ which is ϵ-close to $\bar{f}$ and extends $f|_A$. If f and $\bar{f}$ are real-valued, then g can be take to be real-valued as well.*

Proof. We first construct a map $\hat{f} \in U(X, x_0)$ with $\hat{f}|_A = f|_A$. Set $B = X \setminus W$. Let $\phi : X \to \mathbf{R}$ be defined by $\phi(x) = \frac{d(x,B)}{d(x,A)+d(x,B)}$. Since W is a linear neighborhood, there

is a c such that $d(A \setminus B_r, B \setminus B_r) \geq cr$ for all $r \geq 0$; thus, $\phi \in U(X, x_0)$ by Proposition 2.9. Define $\hat{f} : X \to \mathbf{C}$ by taking $\hat{f}(x) = \phi(x)f(x)$ if $x \in W$, and setting $\hat{f}(x) = 0$ otherwise. Clearly $\hat{f}$ is bounded and $\hat{f}|_A = f|_A$.

Let $x, y \in X$. We consider four cases. First, if $x, y \in W$, then $|\hat{f}(x) - \hat{f}(y)| \cdot \|x\| = |\phi(x)f(x) - \phi(y)f(y)|\|x\| \leq (c_f + \|f\|c_\phi)d(x, y)$, where c_f and c_ϕ are the appropriate constants for f and ϕ. If $x \in W$ and $y \notin W$, then $|\hat{f}(x) - \hat{f}(y)|\|x\| = |\phi(x)f(x)|\|x\| \leq \|f\||\phi(x) - \phi(y)|\|x\| \leq \|f\|c_\phi d(x, y)$. If $x \notin W$ and $y \in W$, then a similar argument shows that $|\hat{f}(x) - \hat{f}(y)|\|x\| \leq \|f\|c_\phi d(x, y)$. Finally, $|\hat{f}(x) - \hat{f}(y)|\|x\| = 0$ when $x, y \notin W$. Thus, we have that $|\hat{f}(x) - \hat{f}(y)|\|x\| \leq (c_f + \|f\|c_\phi)\|x\|$ for all x and y in X. By the remark before Proposition 2.7, $\hat{f}$ is continuous everywhere except possibly at x_0; but W is an open neighborhood containing x_0, so $\hat{f}$ is continuous at x_0 since $\hat{f} = \phi f$ on W. Hence $\hat{f} \in U(X, x_0)$.

Let $\tilde{f} \in U(X, x_0)$ be an $\epsilon/2$-approximation of $\bar{f}$. We consider the function $q = \hat{f} - \bar{f} : X \to \mathbf{C}$. By Proposition 3.3 there is a linear neighborhood W_0 of $A \subset q^{-1}(0)$ such that $W_0 \subset q^{-1}(-\epsilon/2, \epsilon/2)$. Let ψ_1, ψ_2 be defined by

$$\psi_1(x) = \frac{d(x, X \setminus W_0)}{d(x, A) + d(x, X \setminus W_0)} \quad \text{and} \quad \psi_2(x) = \frac{d(x, A)}{d(x, A) + d(x, X \setminus W_0)};$$

so $\psi_1, \psi_2 \in U(X, x_0)$ by Proposition 2.9. We define $g = \psi_1 \hat{f} + \psi_2 \tilde{f}$. Then $g \in U(X, x_0)$ since $U(X, x_0)$ is an algebra. Note that

$$|g - \bar{f}| = |\psi_1 \hat{f} + \psi_2 \tilde{f} - (\psi_1 + \psi_2)\bar{f}| \leq |\psi_1(\hat{f} - \bar{f})| + |\psi_2||\tilde{f} - \bar{f}| \leq \epsilon/2 + \epsilon/2.$$

Finally, it is clear that $g|_A = \hat{f}|_A = f|_A$. $\square$

Proposition 3.5. *Suppose that $u, v \in U(X, x_0)$ are nonnegative, real-valued functions, and $u(x) + v(x) \geq \delta > 0$ for all $x \in X$. Then $\frac{u}{u+v} \in U(X, x_0)$ as well.*

Proof.

$$\left|\frac{u(x)}{u(x)+v(x)} - \frac{u(y)}{u(y)+v(y)}\right|\|x\| \leq \left(\frac{|u(x) - u(y)|}{u(x) + v(x)} + |u(y)|\left|\frac{1}{u(x)+v(x)} - \frac{1}{u(y)+v(y)}\right| \right)\|x\|$$

$$\leq \frac{|u(x) - u(y)|}{u(x) + v(x)}\|x\| + u(y)\frac{|u(x) - u(y)| + |v(x) - v(y)|}{(u(x) + v(x))(u(y) + v(y))}\|x\| \leq \frac{2c_u + c_v}{\delta} d(x, y).$$

$\square$

Proposition 3.5 also holds if we replace $U(X, x_0)$ by $U(X, x_0, \mathbf{R})$ throughout the statement. In what follows, f_i will indicate the ith component of a function $f : X \to \mathbf{R}^{n+1}$.

Lemma 3.6. *Let (X,d) be a proper metric space with basepoint x_0. Let A be a closed subset of X containing the basepoint, and let W be an open linear neighborhood of A. Let Δ denote the standard $n-$simplex. Suppose that $f \in U(W, x_0, \partial\Delta)$ and $g : X \to \partial\Delta$ is a continuous extension of $f|_A$ such that each component of g is a linear Higson function. Then there is an $h \in U(X, x_0, \partial\Delta)$ which extends $f|_A$.*

Proof. Looking at components, by Proposition 3.4 we have that there are $q_i \in U(X, x_0, \mathbf{R})$ such that $q_i|_A = f_i|_A$ and $\|q_i - g_i\| \leq \frac{1}{3(n+1)}$. Thus,

$$|1 - \sum_i |q_i|| = |\sum_i |g_i| - \sum_i |q_i|| \leq \sum_i ||g_i| - |q_i|| \leq \frac{1}{3},$$

and so $\sum_i |q_i(x)| \geq 2/3$ for all $x \in X$. Define $q' : X \to \Delta$ by $q'(x) = \left(\frac{|q_i(x)|}{\sum_j |q_j(x)|}\right)_{i=1}^{n+1}$. This map is well-defined. Also, since $q_i \in U(X, x_0, \mathbf{R})$, we have $|q_i| \in U(X, x_0, \mathbf{R})$, and hence $q'_i \in U(X, x_0, \mathbf{R})$ by Proposition 3.5.

Now, fix $x \in X$; note that there is a j such that $g_j(x) = 0$. So $|q_j(x)| \leq \|q_j - g_j\| \leq \frac{1}{3(n+1)}$. Thus, $\frac{|q_j(x)|}{\sum_k |q_k(x)|} \leq \frac{1}{2(n+1)}$, and so

$$d(q'(x), b) = \sqrt{\sum_{1 \leq i \leq n+1} |\frac{|q_i(x)|}{\sum_k |q_k(x)|} - \frac{1}{n+1}|^2} \geq |\frac{|q_j(x)|}{\sum_k |q_k(x)|} - \frac{1}{n+1}| \geq \frac{1}{2(n+1)},$$

where b is the barycenter of Δ. That is, q' maps X into $\Delta \setminus B_{\frac{1}{2(n+1)}}(b)$. Let $r : \Delta \setminus B_{\frac{1}{2(n+1)}}(b) \to \partial\Delta$ be a $\lambda-$Lipschitz retract onto $\partial\Delta$. Finally, define $h : X \to \partial\Delta$ by $h = r \circ q'$. So $h|_A = r \circ q'|_A = r \circ f|_A = f|_A$ since r is a retract. Also, since $q'_i \in U(X, x_0, \mathbf{R})$ for all i, we have $q' \in U(X, x_0, \Delta \setminus B_{\frac{1}{2(n+1)}}(b))$, and so $h \in U(X, x_0, \partial\Delta)$ by the remarks preceding Proposition 3.2. $\square$

The inequality AN-asdim $X \leq \dim \nu_L X$**.**

We recall that a metric space (X,d) is called *cocompact* if there is a compact subset K of X such that $X = \cup_{\gamma \in \mathrm{Isom}(X)} \gamma(K)$, where $\mathrm{Isom}(X)$ is the set of all isometries of X.

Theorem 3.7. *Let X be a cocompact, connected, proper metric space which has finite asymptotic Assouad-Nagata dimension. Then* $\dim \nu_L X \geq$ AN-asdim X.

Proof. Set $n =$ AN-asdim X. If $n = 0$, the inequality is immediate. We shall henceforth assume that $n > 0$, and so in particular X is not compact. To get a contradiction,

assume that $\dim \nu X \le n-1$. By hypothesis, there is a compact subset K of X such that $X = \cup_{\gamma\in\Gamma}\gamma(K)$, where $\Gamma = \mathrm{Isom}(X)$.

Let $\{\lambda_m\}_{m=1}^{\infty}$ be a sequence of positive numbers, let $\{p_m : X \to P_m\}_m$ be a sequence of maps to $n-$dimensional polyhedra, and let $C > 0$ be a constant such that (1) - (3) of Lemma 3.1 hold. Without loss of generality, one may take $C > \operatorname{diam} K$. Also, passing to subsequences if necessary, we may assume that $\lambda_1 \le 1$ and $\lambda_{i+1} \le \lambda_i/25$. For every i, we take $\Delta_i \subset P_i$ with $D(p_i, \Delta_i) \ge D(p_i)/2$. Let $h_i : \Delta_i \to \Delta$ be an isometry to the standard $n-$simplex.

Let $\gamma_i \in \Gamma$ be an element with $p_i^{-1}(\Delta_i) \cap \gamma_i K \neq \varnothing$; let $x_i \in p_i^{-1}(\Delta_i)\cap\gamma_i K$. Choose y_i with $\|y_i\| = \frac{3C}{\lambda_i}$, and let α_i be such that $y_i \in \alpha_i K$. Define

$$A_i = \alpha_i\gamma_i^{-1}p_i^{-1}(\Delta_i) \quad \text{and} \quad B_i = \alpha_i\gamma_i^{-1}p_i^{-1}(\partial\Delta_i)$$

for $i = 1, 2, \ldots$. We also define $A_0 = B_0 = \{x_0\}$. Note that $\operatorname{diam} A_i \le \frac{C}{\lambda_i}$ when $i \ge 1$.

For $a \in A_i$,

$$d(a, y_i) \le d(a, \alpha_i\gamma_i^{-1}x_i) + d(\alpha_i\gamma_i^{-1}x_i, y_i) \le \frac{C}{\lambda_i} + C \le \frac{2C}{\lambda_i}.$$

So for $a \in A_i$, we have $\frac{3C}{\lambda_i} = \|y_i\| \le \|a\| + d(a, y_i) \le \|a\| + \frac{2C}{\lambda_i}$ and so $d(x_0, A_i) \ge \frac{C}{\lambda_i}$. Also, for $a \in A_i$, $\|a\| \le \|y_i\| + d(y_i, a) \le \frac{5C}{\lambda_i}$, and so $\|A_i\| := \sup_{a\in A_i}\|a\| \le \frac{5C}{\lambda_i}$ for $i \ge 1$. Note that the A_i ($i \ge 0$) are disjoint since

$$\|A_i\| \le \frac{5C}{\lambda_i} \le \frac{C}{5\lambda_{i+1}} < \frac{C}{\lambda_{i+1}} \le d(x_0, A_{i+1})$$

for all $i \ge 1$ and since $x_0 \notin A_i$ for all $i \ge 1$ (because $d(x_0, A_i) > 0$). Set $A = \coprod_{i\ge 0} A_i$ and $B = \coprod_{i\ge 0} B_i$. A and B are closed in X.

Let $q_i = h_i p_i \gamma_i \alpha_i^{-1}|_{A_i}$ for $i \ge 1$ and define $q_0 : A_0 \to \Delta$ by $q_0(x_0) = (1, 0, 0, \ldots, 0)$ (the image does not matter, however). Now take $Q = \coprod_{i\ge 0} q_i : A \to \Delta$. So $\|Q(x) - Q(y)\|\|x\| \le (\operatorname{Lip} q_i)\|A_i\| d(x, y) \le \lambda_i \frac{5C}{\lambda_i} d(x, y)$ when $x, y \in A_i$ for $i \ge 1$. For $x, y \in A_0 = \{x_0\}$, we just have $\|Q(x) - Q(y)\|\|x\| = 0$.

Now suppose that $x \in A_j$ and $y \in A_i$, where $j \neq i$ and $i, j \ge 1$. We first prove this in the case that $j > i$. So $\|Q(x) - Q(y)\|\|x\| \le \sqrt{n+1}\|A_j\| \le \frac{5\sqrt{n+1}C}{\lambda_j}$ and since $\lambda_j \le \lambda_i/25$, we have $d(x, y) \ge d(x_0, A_j) - \|A_i\| \ge \frac{C}{\lambda_j} - \frac{5C}{\lambda_i} \ge \frac{C}{\lambda_j}\left(1 - \frac{1}{5}\right) = \frac{4C}{5\lambda_j}$.

Thus, we have

$$\|Q(x) - Q(y)\|\|x\| \le 5\sqrt{n+1}(\frac{5}{4})d(x, y) = \frac{25}{4}\sqrt{n+1}d(x, y).$$

If $j < i$, then $\|x\| \leq \|y\|$, and so $\|Q(x) - Q(y)\|\|x\| \leq \|Q(y) - Q(x)\|\|y\|$, and thus by appealing to the case just considered, we have $\|Q(x) - Q(y)\| \leq \frac{25}{4}\sqrt{n+1}d(y,x)$. Now suppose $x \in A_j$ and $y \in A_i$, where i and j are distinct and either $j = 0$ or $i = 0$. So either $x = x_0$ or $y = x_0$, and in either case it is easy to check that $\|Q(x) - Q(y)\|\|x\| \leq \sqrt{n+1}d(x,y)$.

Thus, for $c_Q = \max\{5C, \frac{25}{4}\sqrt{n+1}\}$, we have that $\|Q(x) - Q(y)\|\|x\| \leq c_Q d(x,y)$ for all $x, y \in A$. That is, $Q \in U(X, x_0, \Delta)$.

We now construct a map $f : W \to \partial\Delta$, where W is a linear neighborhood of B in A, f extends $Q|_B$, and $f \in U(W, x_0, \partial\Delta)$. Set $\epsilon = \frac{1}{2(n+1)}$ and set $W = Q^{-1}(N_\epsilon(\partial\Delta))$. W is a linear neighborhood of B by Proposition 3.2, and $Q|_W \in U(W, x_0, \Delta)$. Let $b \in \Delta$ be the barycenter and note that $B_\epsilon(b) \cap N_\epsilon(\partial\Delta) = \varnothing$ (neighborhood is taken in Δ). There is a Lipschitz retraction $r : \Delta \setminus B_\epsilon(b) \to \partial\Delta$, and so $f := r \circ Q|_W : W \to \partial\Delta$ is in $U(W, x_0, \partial\Delta)$ by the remarks preceding Proposition 3.2. Clearly $f|_B = Q|_B$.

Also Q, when restricted to $\coprod_{i\geq 0} B_i$, is a map into $\partial\Delta$. Each Q_i extends to a map $\overline{B}$ into $\mathbf{R}$, and so Q extends to a map $G'' : \overline{B} \to \mathbf{R}^{n+1}$. Since G'' is continuous, $G''(\overline{B}) \subset \overline{G''(B)} \subset \overline{\partial\Delta} = \partial\Delta$, and so we view G'' as a map from $\overline{B}$ to $\partial\Delta$. Since νB is closed in νA, where $\dim \nu A \leq \dim \nu X \leq n-1$, we have an extension of $G''|_{\nu B}$ to νA; thus, we have an extension $G' : \overline{B} \cup \nu A \to \partial\Delta$ of G''. Since $\partial\Delta$ is an absolute neighborhood retract, there is a continuous extension $G : V \to \partial\Delta$ of G' to a neighborhood V of $\overline{B} \cup \nu A$ in $\overline{A}$. Thus, there is an m_0 such that $A' := A_0 \coprod(\coprod_{i\geq m_0} A_i)$ is a subset of V; set $g = G|_{A'} : A' \to \partial\Delta$ and note that $g|_{B'} = Q|_{B'}$, where $B' = B_0 \coprod(\coprod_{i\geq m_0} B_i)$. Also, A' is closed in A and so $\overline{A'} \subset V$. So g extends to a continuous map on $\overline{A'} \subset V$, and since the sublinear Higson compactification of A' is homeomorphic to the closure of A' in $\overline{A}$, we have that each component of g is a Higson function on A'.

We now restrict our attention to A'. Note that $W \cap A'$ is a linear neighborhood of $B \cap A' = B'$ in A'. We have that $f|_{A'\cap W} : A' \cap W \to \partial\Delta$ is an element of $U(A' \cap W, x_0, \partial\Delta)$, each component of g is a Higson function, and $g|_{B'} = Q|_{B'} = f|_{B'}$. By lemma 3.6, we have that there is an $h : A' \to \partial\Delta$ such that h extends $f|_{B'}$ and for which there is a c_h such that $\|h(x) - h(y)\|\|x\| \leq c_h d(x,y)$ for all $x, y \in A'$.

We now look at A_i for $i \geq m_0$. So $h|_{A_i} : A_i \to \partial\Delta$ extends $f|_{B_i} = h_i p_i \gamma_i \alpha_i^{-1}|_{B_i}$ and

$$\frac{C}{\lambda_i}\|h(x) - h(y)\| \leq \|h(x) - h(y)\|\|x\| \leq c_h d(x,y)$$

whenenver $x, y \in A_i$. Thus, $h|_{A_i}$ is $\frac{c_h\lambda_i}{C}$–Lipschitz. So

$$\frac{D(p_i)}{2} \leq D(p_i, \Delta_i) \leq \mathrm{Lip}(h_i^{-1} h|_{A_i} \alpha_i \gamma_i^{-1}) = \mathrm{Lip}(h|_{A_i}) \leq \frac{c_h\lambda_i}{C}$$

and thus $\frac{2c_h}{C} \geq \frac{D(p_i)}{\lambda_i} \to \infty$, a contradiction. $\square$

When we apply Theorem 3.7 to the Cayley graph of a finitely generated group we obtain the following.

Corollary 3.8. *For a finitely generated group* Γ *with word metric,* $\dim \nu_L \Gamma \geq \text{AN-asdim}\,\Gamma$ *provided* $\text{AN-asdim}\,\Gamma < \infty$*.*

EXAMPLE. Consider the parabolic region $X = \{(x,y) \in \mathbf{R}^2 : x \geq 0, |y| \leq \sqrt{x}\}$, which we will equip with the (restricted) Euclidean metric. Let $i : [0,\infty) \to X$ be the map $i(x) = (x,0)$. Taking the usual metric on $[0,\infty)$, it is not hard to show that i is a coarse equivalence for the sublinear coarse structures, and hence there is a homeomorphism $\nu_L[0,\infty) \to \nu_L X$. In particular, $\dim \nu_L X = 1$. But $\text{AN-asdim}\, X = 2$. Since X is a connected proper metric space, this shows that we can not drop the requirement in the Theorem that the space be cocompact.

The inequality $\text{AN-asdim}\, X \geq \dim \nu_L X$**.**

If $\mathcal{U}$ is a cover of X and $A \subset X$, then we write $\mathcal{U}_A = \{U \in \mathcal{U} : U \cap A \neq \varnothing\}$.

Lemma 3.9. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be covers of* X*, and suppose that* $\mathcal{U}$ *refines* $\mathcal{V}$*. Let* K *be a subset of* X*. So for each* $U \in \mathcal{U}$ *with* $U \cap (X \setminus K) \neq \varnothing$*, there is a* $V_U \in \mathcal{V}$ *with* $U \subset V_U$*. For* $V \in \mathcal{V}$*, set*

$$V' = [V \cap (X \setminus K)] \bigcup [\bigcup_{U \in \mathcal{U}_{X \setminus K}, V_U = V} U]$$

and define

$$\mathcal{W} = \{U \in \mathcal{U} | U \subset K\} \cup \{V' | V \in \mathcal{V}_{X \setminus K}\}.$$

Then

(1) $\mathcal{W}$ *is a cover of* X*;*
(2) $\text{mult}\,\mathcal{W} \leq n+1$ *if* $\text{mult}\,\mathcal{U} \leq n+1$ *and* $\text{mult}\,\mathcal{V} \leq n+1$*;*
(3) $\mathcal{W}$ *refines* $\mathcal{V}$*;*
(4) $\mathcal{U}$ *refines* $\mathcal{W}$*;*
(5) *if* $W \in \mathcal{W}$ *and* $W \subset K$*, then* $W \in \mathcal{U}$*;*
(6) *if* $\mathcal{U}$ *and* $\mathcal{V}$ *are open covers and* K *is closed in* X*, then* $\mathcal{W}$ *is also an open cover;*
(7) *if* $V \in \mathcal{V}$ *and* $V \cap K = \varnothing$*, then* $V = V' \in \mathcal{W}$*.*

The proof is strightforward. For the above $\mathcal{W}$ we will write

$$\mathcal{W} = \mathcal{U} *_K \mathcal{V}.$$

The following Theorem is a modification of Lemma 2.9 of [DKU].

Theorem 3.10. *Let (X, d) be a proper metric space. Then* $\dim \nu_L X \leq \text{AN-asdim}\, X$.

Proof. We write $\nu X = \nu_L X$ and use $\dot{B}(x, r)$ to denote a closed ball of radius r. Set $n = \text{AN-asdim}\, X$ (if $\text{AN-asdim}\, X = \infty$, the inequality is immediate). As $\{\widetilde{U} \cap \nu X : U \subset X \text{ open}\}$ is a basis for νX, it suffices to prove that each cover of the form $\{\widetilde{U}_i \cap \nu X : 1 \leq i \leq m\}$, where each $U_i \subset X$ is open, admits a finite refinement of multiplicity $\leq n + 1$.

So let $\{\widetilde{U}_i \cap \nu X : 1 \leq i \leq m\}$ be a cover of X, and set $\mathcal{U} = \{U_i : 1 \leq i \leq m\}$. Since $\text{AN-asdim}\, X = n$, there exist $C > 0$ and $r_{-1} > 0$ such that whenever $r \geq r_{-1}$, there is an open cover $\mathcal{U}(r)$ of X satisfying $\operatorname{mult} \mathcal{U}(r) \leq n + 1$, $\operatorname{mesh} \mathcal{U}(r) < Cr$, and $L(\mathcal{U}(r)) > r$. Without loss of generality, we take $C > 1$. Also, there is a $D > 0$ and an $r_{-2} > 0$ such that $L^{\mathcal{U}}(x) \geq D\|x\|$ whenever x is such that $\|x\| \geq r_{-2}$. We may take $D < 1$.

Now, choose $r_0 > \max\{r_{-2}, r_{-1}, 1\}$. Define $r_i = (\frac{C}{D})^i r_0$ for $i \geq 1$. Observe that $r_{i+1} = \frac{C}{D} r_i > r_i > r_0 > 1$. Since $r_i > r_0 > r_{-1}$ for $i \geq 1$, there is a cover $\mathcal{U}_i$ of X such that $\operatorname{mult} \mathcal{U}_i \leq n + 1$, $\operatorname{mesh} \mathcal{U}_i < Cr_i$, and $L(\mathcal{U}_i) > r_i$.

Define $\mathcal{V}_1 = \mathcal{U}_1$, and note that $\operatorname{mesh} \mathcal{V}_1 < Cr_1$. Now, supposing we have defined $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_i$ satisfying $\operatorname{mesh} \mathcal{V}_j < Cr_j < r_{j+1}$ for all $1 \leq j \leq i$, then $\mathcal{V}_i$ refines $\mathcal{U}_{i+1}$, and so we can define

$$\mathcal{V}_{i+1} = \mathcal{V}_i *_{\dot{B}(x_0, 2r_{i+2})} \mathcal{U}_{i+1}.$$

By Lemma 3.9, $\mathcal{V}_{i+1}$ refines $\mathcal{U}_{i+1}$, and so $\operatorname{mesh} \mathcal{V}_{i+1} < Cr_{i+1}$. Thus, we have constructed $\mathcal{V}_i$ for all positive integers i.

Set $\mathcal{V} = \liminf_i \mathcal{V}_i = \cup_s \cap_{t \geq s} \mathcal{V}_t$. We now investigate some properties of $\mathcal{V}$.

Using the definition of $\mathcal{V}_{i+1}$, it is easy to show that if $U \in \mathcal{V}_i$ and $U \cap \dot{B}(x_0, r_{i+2}) \neq \varnothing$, then $U \in \mathcal{V}_{i+1}$. We conclude that $\{U \in \mathcal{V}_i : U \cap \dot{B}(x_0, r_{i+2}) \neq \varnothing\} \subset \mathcal{V}$. As $\mathcal{V}_i$ is a cover of X and hence of $\dot{B}_{r_{i+2}}$, we have that $\mathcal{V}$ covers $\dot{B}_{r_{i+2}}$;as i here is arbitrary, $\mathcal{V}$ covers X.

We now show that if $V \in \mathcal{V}$ and $V \cap \dot{B}_{r_{i+1}} \neq \varnothing$, then $V \in \mathcal{V}_{i-1}$. First suppose that $V \in \mathcal{V}_i$ and $V \cap \dot{B}_{r_{i+1}} \neq \varnothing$; then $\operatorname{mesh} \mathcal{V}_i < Cr_i < r_{i+1}$ implies that $V \subset \dot{B}_{2r_{i+1}}$ and so $V \in \mathcal{V}_{i-1}$ by (5) of the lemma. Now suppose $V \in \mathcal{V}$. This means there is an $s \geq i - 1$ such that $V \in \mathcal{V}_s$. Applying the result we just found and proceeding inductively, one can show that $V \in \mathcal{V}_j$ for all j such that $i - 1 \leq j \leq s$.

We show that $\mathcal{V}_i$ refines $\mathcal{V}$ for all $i \geq 1$. We know by (4) of the lemma that $\mathcal{V}_i$ refines $\mathcal{V}_{i+1}$ for all $i \geq 1$. Fixing i, let $V \in \mathcal{V}_i$. Choose $j \geq i$ such that $V \cap \dot{B}_{r_{j+2}} \neq \varnothing$. As $\mathcal{V}_i$ refines $\mathcal{V}_j$, there is a $U \in \mathcal{V}_j$ such that $V \subset U$. Also, $U \cap \dot{B}_{r_{j+2}} \supset V \cap \dot{B}_{r_{j+2}} \neq \varnothing$. Thus, $V \subset U \in \mathcal{V}$. So $\mathcal{V}_i$ refines $\mathcal{V}$.

Since each $\mathcal{V}_i$ has multiplicity $\leq n + 1$ for each i, it is clear from the definition that $\mathcal{V}$ has multiplicity $\leq n + 1$.

Set $\mathcal{W} = \mathcal{V}_{X \setminus \dot{B}_{r_2}}$. We have that $(\mathcal{V}_i)_{X \setminus \dot{B}_{r_2}}$ refines $\mathcal{W}$.

We show that $\mathcal{W}$ refines $\mathcal{U}$. Let $W \in \mathcal{W}$. So there is an $x \in W$ such that $\|x\| > r_2$. Take $i = \max\{j : \|x\| > r_j\}$, and note that $i \geq 2$. Thus, $\|x\| \leq r_{i+1}$, or $x \in \dot{B}_{r_{i+1}}$. Hence $W \cap \dot{B}_{r_{i+1}} \neq \varnothing$. Since $W \in \mathcal{V}$, we have $W \in \mathcal{V}_{i-1}$. So

$$\operatorname{diam} W < Cr_{i-1} = D(\frac{C}{D})r_{i-1} = Dr_i \leq D\|x\| \leq L^{\mathcal{U}}(x),$$

and so there is a $U \in \mathcal{U}$ with $W \subset U$.

We show that $L^{\mathcal{W}} : X \to [0, \infty)$ is at least linear. Set $a = 3r_2$, and let x be an element of X with $\|x\| \geq a = 3r_2$. Set $i = \max\{j : 3r_{j+1} \leq \|x\|\}$, and note that $i \geq 1$ and $3r_{i+1} \leq \|x\| < 3r_{i+2}$. As $L(\mathcal{U}_i) > r_i$, there is a $U \in \mathcal{U}_i$ such that $B(x, r_i) \subset U$. Since $\|x\| \geq 3r_{i+1}$ and $\operatorname{diam} U \leq \operatorname{mesh} \mathcal{U}_i < Cr_i < r_{i+1}$, we have that $U \subset X \setminus \dot{B}(x_0, 2r_{i+1})$. By definition of $\mathcal{V}_i$, and by (7) of the lemma, we have that $U \in \mathcal{V}_i$. In fact, as $\|x\| > r_2$, we have $U \in (\mathcal{V}_i)_{X \setminus \dot{B}_{r_2}}$. Since $(\mathcal{V}_i)_{X \setminus \dot{B}_{r_2}}$ refines $\mathcal{W}$, we have that

$$L^{\mathcal{W}}(x) \geq d(x, X \setminus U) \geq r_i.$$

But $\|x\| < 3r_{i+2} = 3\frac{C^2}{D^2} r_i$, so $L^{\mathcal{W}}(x) > \frac{D^2}{3C^2}\|x\|$. Therefore, $L^{\mathcal{W}}$ is at least linear.

To summarize, $\mathcal{W}$ covers $X \setminus \dot{B}_{r_2}$, $\operatorname{mult} \mathcal{W} \leq n+1$, $\mathcal{W}$ refines $\mathcal{U}$, and $L^{\mathcal{W}}$ is at least linear. Thus, for $W \in \mathcal{W}$, there is a $U_W \in \mathcal{U}$ for which $U \subset U_W$. So for each $1 \leq i \leq m$, we define $W_i = \cup_{U_W = U_i} W$.

Now set $\mathcal{W}' = \{W_i\}$. It follows that $\mathcal{W}$ refines $\mathcal{W}'$ and $\mathcal{W}'$ has multiplicity $\leq n+1$. Thus, $L^{\mathcal{W}'} \geq L^{\mathcal{W}}$ and hence $L^{\mathcal{W}'}$ is at least linear. As a consequence, if we define $\widetilde{\mathcal{W}'} = \{\widetilde{W}_i \cap \nu X\}$, we have that $\widetilde{\mathcal{W}'}$ is a cover of νX. Since $\mathcal{W}'$ refines $\mathcal{U}$, we have that $\widetilde{\mathcal{W}'}$ refines $\widetilde{\mathcal{U}}$. Finally, as $\mathcal{W}'$ has multiplicity $\leq n+1$, so does $\widetilde{\mathcal{W}'}$. $\square$

Corollary 3.11. *For a cocompact connected proper metric space,* $\text{AN-asdim}\, X = \dim \nu_L X$ *provided* $\text{AN-asdim}\, X < \infty$*.*

In particular, we conclude that $\text{AN-asdim}\,\Gamma = \dim \nu_L \Gamma$ for a finitely generated group Γ with $\text{AN-asdim}\,\Gamma < \infty$.

§4 Morita type theorem for Assouad-Nagata dimension

Let $K \subset S^n$ be a compact set in the unit sphere $S^n \subset \mathbf{R}^{n+1}$. The open cone OK by definition is the union of rays through K issuing from the origin with the metric restricted from $\mathbf{R}^{n+1}$. The open cone admits a natural compactification by K, which we will denote by $\overline{OK}$.

Proposition 4.1. *The natural compactification of the open cone OK is dominated by the sublinear compactification.*

Proof. It suffices to show that if two sets $A, B \subset OK$ do not intersect at the cone boundary then they are divergent in the sublinear coarse structure, but this is obvious. $\square$

Lemma 4.2. *Let X be a proper metric space. Then there is an embedding*

$$\nu_L X \times [0,1] \to \nu_L(X \times \mathbf{R}_+).$$

Proof. Let $d_0 : X \to \mathbf{R}_+$ be the distance function to the base point $x_0 \in X$, i.e., $d_0(x) = \|x\|$. Then the map $d_0 \times 1 : X \times \mathbf{R}_+ \to \mathbf{R}_+ \times \mathbf{R}_+$ is a coarse morphism for the sublinear coarse structures and hence is extendible to the sublinear Higson compactifications:

$$\gamma = \overline{(d_0 \times 1)} : h_L(X \times \mathbf{R}_+) \to h_L(\mathbf{R}_+ \times \mathbf{R}_+).$$

Let K be the arc in $S^1 \subset \mathbf{R}^2$ from 0 to $\pi/4$. In view of Proposition 4.1 we have a natural map $\phi' : h_L OK \to \overline{OK}$. Also, tan can be defined on $\overline{OK} \setminus \{0\}$. Let $W = (d_0 \times 1)^{-1}(OK)$. Define $\phi : h_L W \setminus \{(x_0, 0)\} \to [0,1]$ as the composition $(\tan) \circ \phi' \circ \gamma$ restricted to $h_L W \setminus \{(x_0, 0)\}$. Note that, by Corollary 2.6, $\nu_L W \subset \nu_L(X \times \mathbf{R}_+)$.

The restriction of the projection $X \times \mathbf{R}_+ \to X$ to W is a coarse morphism and hence it defines a map $\psi : \nu_L W \to \nu_L X$. We show that the map $\Phi = (\phi|_{\nu_L W}, \psi) : \nu_L W \to [0,1] \times \nu_L X$ is a homeomorphism.

First we note that for every $t \in K$ the preimage

$$X_t = (d_0 \times 1)^{-1}(Ot) = \{(x, \|x\| \tan(t)) \mid x \in X\} \subset X \times \mathbf{R}_+$$

is coarsely equivalent to X. This implies that the map Φ takes $\nu_L X_t$ onto $\tan(t) \times \nu_L X$ homeomorphically. Thus, Φ is onto.

It remains to show that $\Phi^{-1}(\tan(t) \times \nu_L X) = \nu_L X_t$, or equivalently $\phi|^{-1}_{\nu_L W}(\tan(t)) = \nu_L X_t$. Let $z \in \phi|^{-1}_{\nu_L W}(\tan(t))$ and $z \notin \nu_L X_t$. Using proposition 1.2, one can choose a subset A of W with $z \in \overline{A}$ and $\nu_L W \cap \overline{A} \cap \overline{X_t} = \varnothing$ (here bar denotes closure in $h_L W$). So for some $C, r_0 > 0$, $L(x) := \max\{d(x, A), d(x, X_t)\} \geq C\|x\|$ when $x \in W$ with $\|x\| \geq r_0$. In particular, for $(y, s) \in A$ with $\|(y, s)\| \geq r_0$, have

$$|s - \|y\| \tan(t)| = d((y, s), (y, \|y\| \tan(t))) \geq L(y, s) \geq C\|(y, s)\|,$$

whence $|(s/\|y\|) - \tan(t)| \geq C$. But $\phi(y, s) = s/\|y\|$. It follows that $|\phi(z) - \tan(t)| \geq C$ and so $\phi(z) \neq \tan(t)$, a contradiction.

Theorem 4.3. *Let X be cocompact connected proper metric space. Then*

$$\text{AN-asdim}(X \times \mathbf{R}) = \text{AN-asdim}\, X + 1.$$

Proof. By Theorem 3.10, Lemma 4.2, the classical Morita theorem, and by Theorem 3.7 we obtain

$$\text{AN-asdim}(X \times \mathbf{R}) \geq \dim \nu_L(X \times \mathbf{R}) \geq \dim(\nu_L X \times [0,1]) = \dim \nu_L X + 1 \geq \text{AN-asdim}\, X + 1.$$

The opposite inequality is obvious. □

§5 Embedding of asymptotic cones into the sublinear Higson corona

We recall the definition of the asymptotic cone $cone_\omega(X)$ of a metric space with the base point $x_0 \in X$ with respect to a nonprincipal ultrafilter ω on $\mathbf{N}$ [Gr],[Ro2]. On the sequences of points $\{x_n\}$ with $\|x_n\| \leq Cn$ for some C, we define an equivalence relation

$$\{x_n\} \sim \{y_n\} \Leftrightarrow \lim_\omega d(x_n, y_n)/n = 0.$$

We denote by $[\{x_n\}]$ the equivalence class of $\{x_n\}$. The space $cone_\omega(X)$ is the set of equivalence classes $[\{x_n\}]$ with the metric $d_\omega([\{x_n\}], [\{y_n\}]) = \lim_\omega d(x_n, y_n)/n$. We note that the space $cone_\omega(X)$ does not depend on the choice of the base point.

We note that any two constant sequences are equivalent and denote by $[x_0]$ the base point they define in $cone_\omega(X)$.

We call a non-principal ultrafilter ω on $\mathbf{N}$ *exponential* if it contains the image of a function $f : \mathbf{N} \to \mathbf{N}$ satisfying the inequality $f(n+1) \geq af(n)$ for some $a > 1$ and for all but finitely many n. In view of the inequality $f(n+k) \geq a^k f(n)$, the number a for a given exponential ultrafilter ω can be taken arbitrarily large.

Theorem 5.1. *For every exponential ultrafilter ω on $\mathbf{N}$ and for every proper metric space X, there is an injective continuous map $\xi : cone_\omega(X) \setminus [x_0] \to \nu_L X$.*

Moreover, the restriction of ξ to the D-annulus $A_D = \dot{B}_D([x_0]) \setminus B_{1/D}([x_0])$ is an embedding for all $D > 0$.

Proof. We define

$$\xi([\{x_n\}]) = \nu_L X \cap \bigcap_{S \in \omega} \overline{\{x_n \mid n \in S\}}$$

where the closure is taken in the sublinear Higson compactification $h_L X$.

First we show that for every point $[\{x_n\}] \neq [x_0]$ the above intersection is nonempty. Indeed for every finite family of sets $S_1, \dots S_m \in \omega$ we have

$$\nu_L X \cap \bigcap_{i=1}^{m} \overline{\{x_n \mid n \in S_i\}} \supset \nu_L X \cap \overline{\{x_n \mid n \in \cap_{i=1}^{k} S_i\}} \neq \emptyset.$$

Then by the compactness of $\nu_L X$ we obtain the required result.

Next we show that this intersection is a point. Assume that $x, y \in \nu_L X \cap \bigcap_{S \in \omega} \overline{\{x_n \mid n \in S\}}$, $x \neq y$. Let U and V be a disjoint neighborhoods of x and y in $h_L X$ (in the sense that U, V might not be open) such that $U \cap V = \emptyset$ and $h_L X = U \cup V$ in $h_L X$. Let $S_x = \{n \in \mathbf{N} \mid x_n \in U\}$ and $S_y = \{n \in \mathbf{N} \mid x_n \in V\}$. By the definition of ultrafilter, one and only one of these sets belongs to ω, say $S_x \in \omega$. Hence $y \in \overline{\{x_n \mid n \in S_x\}}$. Therefore $y \in \bar{U}$ which contradicts the fact that V is a neighborhood of y and V and U are disjoint.

Next we show that if $\{x_n\} \sim \{y_n\}$, then $\xi([\{x_n\}]) = \xi([\{y_n\}])$. Assume the contrary:

$$\nu_L X \cap \bigcap_{S \in \omega} \overline{\{x_n \mid n \in S\}} \cap \bigcap_{S \in \omega} \overline{\{y_n \mid n \in S\}} = \emptyset.$$

Then

$$\nu_L X \cap \bigcap_{S \in \omega} (\overline{\{x_n \mid n \in S\}} \cap \overline{\{y_n \mid n \in S\}}) = \emptyset.$$

By compactness of $\nu_L X$ there is $\bar{S} \in \omega$ such that

$$\nu_L X \cap (\overline{\{x_n \mid n \in \bar{S}\}} \cap \overline{\{y_n \mid n \in \bar{S}\}}) = \emptyset.$$

By Lemma 2.3 there is $c > 0$ such that $d(x_n, \{y_k\}) \geq c\|x_n\|$ for large enough n. Since $[\{x_n\}] \neq [x_0]$ there is $a > 0$ and $S_0 \in \omega$ such that $\|x_n\| \geq an$ for $n \in S_0$. Then for $n \in S_0 \cap \bar{S}$ we have $d(x_n, y_n) \geq acn$. Since $\{x_n\} \sim \{y_n\}$ there is $S_1 \in \omega$ such that $d(x_n, y_n)/n < ac/2$ for $n \in S_1$. Then for $n \in S_0 \cap S_1 \cap \bar{S}$ we obtain a contradiction: $ac/2 > d(x_n, y_n)/n \geq ac$.

Next we show that ξ is continuous. Let $\lim_{k \to \infty} [\{x_n^k\}] = [\{x_n\}]$. Let $\xi([\{x_n\}]) \in U$, where U is an open neighborhood in $h_L X$. Then for some $S \in \omega$ we have $\{x_n \mid n \in S\} \subset U$. By Lemma 2.3 and the fact that $[\{x_n\}] \neq [x_0]$ there is $c > 0$ such that $d(x_n, X \setminus U') \geq cn$ for some $S' \in \omega$ and some U' whose closure is contained in U, $\bar{U}' \subset U$. Let $\lim_\omega d(x_n^k, x_n)/n = \delta^k$. Then $\delta^k \to 0$. So, for large enough k we have $\delta_k < c/4$. Then there is $S_k \in \omega$ such that $d(x_n^k, x_n)/n < 2\delta^k$ for $n \in S_k$. Then for $n \in S' \cap S_k$, where k is large, we obtain $x_n^k \in U'$. Hence $\overline{\{x_n^k \mid n \in S' \cap S_k\}} \subset U$ for large enough k. Therefore $\xi([\{x_n^k\}]) \in U$ for sufficiently large k.

We show that ξ is injective. Assume that $\xi([\{x_n\}]) = \xi([\{y_n\}])$ and $[\{x_n\}] \neq [\{y_n\}]$. The latter implies that there is $\epsilon > 0$ such that $S = \{n \mid d(x_n, y_n)/n > \epsilon\} \in \omega$. We may assume that $|\|x_n\| - D_1 n| < \delta n$ and $|\|y_n\| - D_2 n| < \delta n$ for $n \in S$ for some small δ where $D_1 = \|[\{x_n\}]\|$ and $D_2 = \|[\{y_n\}]\|$. We also may assume that $S \subset im(f)$ where f is from the definition of the exponential ultrafilter with $a \geq \max\{\frac{2D_1+D_2+3\delta}{D_2-\delta}, \frac{2D_2+D_1+3\delta}{D_1-\delta}\}$. We claim that $d(x_n, \{y_m \mid m \in S\}) = d(x_n, y_n)$ and $d(y_n, \{x_m \mid m \in S\}) = d(x_n, y_n)$ for $n \in S$. Indeed, for $m > n$ we have $d(x_n, y_m) \geq \|y_m\| - \|x_n\| \geq (D_2 - \delta)m - (D_1 + \delta)n = (D_2 - \delta)f(k+l) - (D_1 + \delta)f(k)$, where k and $l \geq 1$ are chosen such that $f(k) = n$ and $f(k+l) = m$. Then $d(x_n, y_m) \geq (a^l(D_2 - \delta) - (D_1 + \delta))f(k) \geq ((D_1 + \delta) + (D_2 + \delta))n \geq \|x_n\| + \|y_n\| \geq d(x_n, y_n)$. A similar argument works for the case $m < n$.

By Lemma 2.3, the sets $\{x_n \mid n \in S\}$ and $\{y_n \mid n \in S\}$ diverge in the space $Z = \cup_{n \in S}\{x_n, y_n\}$ and hence in X. Then $\nu_L X \cap \overline{\{x_n \mid n \in S\}} \cap \overline{\{y_n \mid n \in S\}} = \emptyset$, a contradiction.

To complete the proof it suffices to show that ξ restricted to the D-annulus is open.

For every $[\{x_n\}] \in cone_\omega(X)$ with $\|[\{x_n\}]\| = c \leq D$ and for every R we construct an open set $U \subset X$ such that $\widetilde{U}$ contains $\xi([\{x_n\}])$ and $\widetilde{U} \cap \xi(A_D) \subset \xi(B_R([\{x_n\}]))$. Let $\lambda \geq \max\{4D/c, 2cD\}$. We consider $f : \mathbf{N} \to \mathbf{N}$ with the property $f(n+1) \geq (1+\lambda)f(n)$ and $S = Im(f) \in \omega$. Additionally we may assume that $|\|x_n\| - cn| < \epsilon n$ for $n \in S$ for some small ϵ ($\epsilon < \min\{c/4, 1/(4D), 1/2\}$). We define $U = \cup_{n \in S} B_{\alpha\|x_n\|}(x_n)$ for $\alpha < \min\{1/2, R/(2c)\}$.

Let $\xi([\{z_n\}]) \in \widetilde{U}$ and $\|[\{z_n\}]\| = d$, $1/D \leq d \leq D$. The latter implies that on some $S' \in \omega$, $S' \subset S$ we have $|\|z_k\| - dk| < \epsilon k$. The former implies that for some $S' \in \omega$ and for every $k \in S'$ there exists $n_k \in S$ such that $d(z_k, x_{n_k}) < \alpha\|x_{n_k}\|$. We may assume that in both cases we have the same S' and $S' \subset S$.

We claim that $n_k = k$. Assume that $n_k > k$. Then the triangle inequality

$$(d + \epsilon)k > \|z_k\| \geq \|x_{n_k}\|(1 - \alpha) \geq (c - \epsilon)(1 - \alpha)n_k.$$

implies that $n_k - k < \frac{4d}{c}k \leq \frac{4D}{c}k \leq \lambda k$. Let $k = f(l)$. Then $f(l+s) = n_k$ for some $s \geq 1$ and we obtain a contradiction: $f(l+1) - f(l) \leq f(l+s) - f(l) < \lambda f(l)$. If we assume that $n_k < k$, then the chain of inequalities

$$(1 + \alpha)(c + \epsilon)n_k > (1 + \alpha)\|x_{n_k}\| \geq \|z_k\| > (d - \epsilon)k$$

implies that $k - n_k < (2c/d)n_k \leq 2cDn_k \leq \lambda n_k$. If $n_k = f(l)$, then $k = f(l+s)$ for $s \geq 1$. Then we obtain the same contradiction: $f(l+1) - f(l) \leq f(l+s) - f(l) < \lambda f(l)$.

By the construction, $d(z_k, x_k) < \alpha\|x_k\| \leq \alpha(c + \epsilon)k \leq 2\alpha ck < Rk$. Hence $[\{z_k\}] \in B_R([\{x_k\}])$. $\square$

We recall that a topological space Y is called *strongly paracompact* if every open cover of Y admits a star-finite refinement. It is known that a separable metric space is strongly paracompact and not all metric spaces are strongly paracompact [En].

Corollary 5.2. *For a proper metric space X and an exponential ultrafilter ω, and for every separable $Y \subset cone_\omega(X)$,*

$$\dim Y \leq \text{AN-asdim}\, X.$$

Proof. We present $Y = \cup_n(\bar{A}_n \cap Y) \cup ([x_0] \cap Y)$. Being separable metric spaces, the $\bar{A}_n \cap Y$ are strongly paracompact. By 3.1.23 of [En], $\dim Z' \leq \dim Z$ for a strongly paracompact subspace $Z' \subset Z$. In view of Theorems 5.1 and 3.11 we obtain $\dim(\bar{A}_n \cap Y) = \dim \xi(\bar{A}_n \cap Y) \leq \dim \nu_L X \leq \text{AN-asdim}(X)$. The countable union theorem completes the proof. $\square$

QUESTION. Is it true that for a finitely generated group G and an ultrafilter ω,

$$\dim cone_\omega G = \sup\{\dim Y \mid Y \subset cone_\omega G\},$$

where the supremum is taken over all separable subspaces Y?

The answer is negative if one replaces the asymptotic cone by an arbitrary metric space.

REMARK. Most likely Theorem 5.1 does not hold for general ultrafilters. It seems to us that to have harmony here one should consider the sublinear Higson corona $\nu_L^\omega X$ associated with an ultrafilter ω. Also one can define an Assouad-Nagata dimension $\text{AN-asdim}_\omega X$ depending on ω and repeat all basic steps of this paper to obtain the inequality $\dim cone_\omega(X) \leq \text{AN-asdim}_\omega X$.

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